ON EXTREMUMS OF SUMS OF POWERED DISTANCES TO A FINITE SET OF POINTS

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Abstract. In this paper we investigate the extremal properties of the sum

$$\sum_{i=1}^{n} |MA_i|^{\lambda},$$

where A_i are vertices of a regular simplex, a cross-polytope (orthoplex) or a cube and M varies on a sphere concentric to the sphere circumscribed around one of the given polytopes. We give full characterization for which points on Γ the extremal values of the sum are obtained in terms of λ. In the case of the regular dodecahedron and icosahedron in $\mathbb{R}^3$ we obtain results for which values of λ the corresponding sum is independent of the position of M on Γ. We use elementary analytic and purely geometric methods.

1. Introduction

We investigate problems of the extremality of sums of the kind

$$\sum_{i=1}^{n} |MA_i|^{\lambda}, \tag{1.1}$$

where A_i are vertices of a regular simplex, a cross-polytope (orthoplex) or a cube and M varies on a sphere concentric to the sphere circumscribed around one of the given polytopes.

Such questions arise frequently as physical problems. The case $\lambda = -1$ is equivalent to finding the points on the sphere with maximal and minimal potential. There is an extensive list of articles on the topic of finding extremal point configurations on the sphere with respect to some potential function [2], [3], and an experimental approach was presented in [4]. Later the question arised to find extremums on the sphere with respect to some potential function and fixed base points.

This proves to be a difficult task, and no complete classification exists. Even in the case when the potential function is the powered Euclidean distance, which we investigate in this article, extremal points vary according to the paramether λ.

The results given in the currennt article for the case $\lambda = -1$ can also be used to obtain some restrictions about the zeros of the gradient of the electric field, generated by point-charges at the points A_i.

We also investigate the values of λ for which (1.1) is independent of the position of the point M on the sphere. This values correspond to codes with certain strenght. Early works on this problem characterize the values of λ for which the corresponding sum is independent of the movement of M for the vertices of the regular polygon. Later papers consider the more general problem for multidimensional polytopes.

Of interest is also the situation in which the parameter λ is fixed, but the base points are variable. This gives rise to a family of so-called polarization inequalities [6], [5]. In [5], the problem is considered for arbitrary n points on the unit circle, M also on the circle and $\lambda = -2$ and this case is fully resolved. In [6] the the question is considered in the plane, where the potential is an abitrary convex function on the geodesic distances between points on the circle.

The planar case of the problem we investigate here is consudered in [8], where a characterization of the extremal values of the sum (1.1) is given for arbitrary three points and $\lambda \in (0; 2)$.

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The author of that publication also obtains results about the function when the base points are vertices of the regular n-gon and $\lambda \in (0; 2n)$. Later this results are improved and full characterization has been given in the case of three base points or when the base points are vertices of a regular polygon in [7].

Previous results also include [9], where partial results are obtained when the base points are the vertices of the regular simplex, cross-polytope and cube when $\lambda \in (0; 2)$, using an integral transform from metric geometry. In this paper we give full characterization and expand those result to all values of λ. We give full characterization of the sum (1.1) when A_i are vertices of a regular simplex, a cross-polytope or a cube and M is a point on a sphere concentric to the sphere circumscribed around the given polytopes.

In the case of the regular dodecahedron and icosahedron in $\mathbb{R}^3$ we obtain results for which values of λ the corresponding sum is independent of the position of M on Γ.

We begin with consideration of the planar case, as we will later use results from this section.

2. Planar case

We begin with a few results for a planar equivalent to the given problem, as this will be the basis to continue in higher dimensions.

Let $A_i, i = 1, \dots, n$, be the vertices of a regular n-gon inscribed in the unit circle. Now assume that Γ is a circle concentric to the circumscribed circle. Put $B_i = OA_i \bigcap \Gamma$, where O is the center of the n-gon.

Let $X \in \Gamma$ be a variable point and

$$R_n(X, \lambda) = \sum_{i=1}^{n} (\sqrt{|XA_i|^2 + h})^{\lambda},$$

where $h \geq 0$ i some fixed real number. In a previous article [7] the authors have given full characterization of $R_n(X, \lambda)$ in terms of λ when $h = 0$. It is easy and straightforward to modify the proof given there to verify that the following Theorem holds for all $h \geq 0$.

Theorem 1.

(1) $\lambda < 0$. *The minimum of $R_n(X, \lambda)$ is achieved when X bisects the arc between consecutive vertices of $B_1 \dots B_n$ and the maximum when $X \equiv B_i$. This function is not bounded in the case when Γ is the circumscribed circle around $A_1 \dots A_n$ and $h = 0$.($X \to B_i$ for some i).*

(2) $0 \leq \lambda < 2n$. *If λ is an even integer, then $R_n(X, \lambda)$ is independent of the position of X on Γ.*
Otherwise let m be such an integer, that $2m \leq \lambda \leq 2m + 2$.
If m is even (odd) then $R_n(X, \lambda)$ is maximal (minimal) if and only if X bisects the arc between consecutive vertices of $B_1 \dots B_n$. Moreover $R_n(X, \lambda)$ is minimal (maximal) if and only if $M \equiv B_i$.

(3) $2n \leq \lambda$. *The maximum (minimum) of $R_n(X, \lambda)$ is obtained when X coincides with one of the vertices of the $B_1 \dots B_n$ when n is even (odd) and the minimum (maximum) is achieved when X bisects the arc between consecutive vertices when n is even (odd).*

Remark 1. The term h can be interpreted in the following way. Assume that Γ and the points A_i belong to different two-dimensional planes at distance $\sqrt{h}$. Let the extremal value of $\sum_{i=1}^{n} |MA_i|^{\lambda}$ be attained at some point M_0. Then $\sum_{i=1}^{n} |M_0'A_i|^{\lambda}$ is an extremal value of $\sum_{i=1}^{n} |M'A_i|^{\lambda}$, where M' is the projection of M in the plane containing A_i.

Remark 2. It can be proved that $\sum_{i=1}^{n} |PA_i|^{2k} = \sum_{j=0} (R + r)^{k-2j} R^k r^k \binom{k}{2j}\binom{2j}{j}$ where R and r are the radii of Γ and the circumscribed circle around the polygon when $k \in \{2, 4, \dots, 2n-2\}$. When $R = r$ this sum transforms to $\binom{2k}{k} nr^{2m}$. The proof is straightforward from the use of complex numbers and is irrelevant to the current work, so we shall not present it here.

It has been proved that $\{2, 4, \dots, 2(n-1)\}$ are the only integers j for which the sum $\sum_{i=1}^{n} |PA_i|^j$ is independent of the position of P on Γ. By Theorem 1 it follows that these are the only real values of λ with this property.

As it turns out this is a characteristic property of the regular polygon.

Theorem 2. *Given n different points $A_1, A_2, \dots, A_n$ in the plane and a circle Γ such that $\sum_{i=1}^n |PA_i|^{2k}$ is independent of the position of P on Γ for $k \in \{1, 2, \dots, n-1\}$. Then these points are the vertices of a regular polygon, inscribed in a circle concentric to Γ.*

Proof. We consider the problem in the complex plane and we assume Γ to be the unit circle. We assign complex numbers $a_1, \dots a_n$ to $A_1, \dots, A_n$ respectively. Using the complex polynomial method we obtain

$$\sum_{i=1}^{n} |x - a_i|^{2k} = \sum_{i=1}^{n} (x - a_i)^k (\overline{x - a_i})^k = \sum_{i=1}^{n} (x - a_i)^k (\frac{1}{x} - \overline{a_i})^k = c.$$

After multiplying out we obtain

$$(x - a_i)^k (\frac{1}{x} - \overline{a_i})^k = \sum_{j=-k}^{k} c_{ij} x^j = P_i(x).$$

Now we have

$$\sum_{i=1}^{n} |x - a_i|^{2k} = \sum_{i=1}^{n} P_i(x) = \sum_{j=-k}^{k} \big(\sum_{i=1}^{n} c_{ij}\big) x^j = c$$

and after multiplying by x^k we get:

$$\sum_{j=-k}^{k} \big(\sum_{i=1}^{n} c_{ij}\big) x^{j+k} - c x^k = 0.$$

This polynomial has infinitely many zeros (all x with $|x| = 1$) and so it is identically zero. In particular, we have that $\left(\sum_{i=1}^n c_{ik}\right) = \sum_{i=1}^n \overline{a_i}^k = 0$. It follows that $\sum_{i=1}^n a_i^k = 0$ for all $k \in \{1, \dots, n-1\}$. Using Newton's identities this implies the desired result. □

We may pose the following

Conjecture 1. *Given n different points $A_1, A_2, \dots, A_n$ in the plane and a circle Γ such that $\sum_{i=1}^n |PA_i|^{2n-2}$ is a constant function of $P \in \Gamma$. Then these points are the vertices of a regular polygon, inscribed in a circle concentric to Γ.*

For $n = 2$ this conjecture is trivial. To conform it for $n = 3$, we shall need the following proposition (which is also of independent interest).

Proposition 1. *Let $A_1, \dots, A_n$ be points in the plane, which belong to a circle T. Assume that Γ is a circle, concentric to T, such that $\sum_{i=1}^n |XA_i|^{2k}$, where $k > \left[\frac{n}{2}\right]$, is independent of the position of X on Γ, then $A_1, \dots, A_n$ are the vertices of a regular $n-gon$.*

Proof. We assign complex numbers a_i to the points A_i and x to X. After a dilatation and a rotation we may assume that T is the unit circle and that $\prod_{i=1}^n a_i = 1$. We now have

$$\sum_{i=1}^{n} |x - a_i|^{2k} = \sum_{i=1}^{n} (x - a_i)^k (\overline{x} - \overline{a_i})^k = const.$$

Using the same approach as before and $|a_i| = |a_j|, i, j \in \{1, \dots, n\}$ we obtain that $\sum_{i=1}^n a_i^t = 0$ for $t = 1, \dots, k$. If $k \geq n-1$ we easily obtain that $A_1, \dots, A_n$ are the vertices of a regular n-gon, but from here it follows that $\sum_{i=1}^n a_i^t \neq 0$, when t is a multiple of n and hence $k = n-1$. Now if $\left[\frac{n}{2}\right] \leq k < n-1$ we have by Newton's identities that $e_i(a_1, \dots, a_n) = 0$ for $i = 1, \dots, k$. Now from $|a_i| = 1$ and $\prod_{i=1}^n a_i = 1$ we have that if $A \subset \{1, \dots n\}$, then

$$\overline{\prod_{i \in A} a_i} = \prod_{i \in A} \overline{a_i} = \prod_{i \in A} \frac{1}{a_i} = \frac{1}{\prod_{i \in A} a_i} = \prod_{i \notin A} a_i.$$

Thus we have $\overline{e_i(a_1, \dots, a_n)} = e_{n-i}$ and hence $e_i(a_1, \dots, a_n) = 0$ for $i = 1, \dots, n-1$ as $\left[\frac{n}{2}\right] \leq k$. The conclusion easily follows. □

Proposition 2. *Let A_1, A_2 and A_3 be three different points in the plane. Assume that there exists a circle Γ such that $|XA_1|^4+|XA_2|^4+|XA_3|^4$ is independent of the position of X on Γ. Then A_1, A_2 and A_3 are the vertices of an equilateral triangle.*

Proof. We shall use complex numbers. After a dilatation we can consider Γ to be the unit circle. We assign complex numbers a_i to the points A_i and x to X. Assume that some of a_i is zero, say $a_1=0$. This is equivalent to $|XA_2|^4+|XA_3|^4$ is independent of the position of X on Γ, but this is not an equation of a circle, unless $A_1\equiv A_2\equiv A_3$, which is not the case. Now we have that

$$\sum_{i=1}^{3}|XA_i|^4=\sum_{i=1}^{3}(x-a_1)^2(\overline{x}-\overline{a_1})^2=c.$$

Then $x^2\sum_{i=1}^{3}(x-a_1)^2(\overline{x}-\overline{a_1})^2-x^2c=0$. As this polynomial in x is zero for all $|x|=1$ we have that it is identically zero. Hence $a_1^2+a_2^2+a_3^2=0$ and $(1+|a_1|^2)a_1+(1+|a_2|^2)a_2+(1+|a_3|^2)a_3=0$. Put $1+|a_1|^2=p, 1+|a_2|^2=q, 1+|a_3|^2=r$. Thus we have that $pa_1+qa_2+ra_3=0$ and then

$$a_1^2+a_2^2+\Big(\frac{pa_1+qa_2}{r}\Big)^2=0.$$

This is equivalent to $a_1^2(r^2+p^2)+a_2^2(r^2+q^2)+2pqa_1b_2=0$. After dividing by $a_2^2\neq 0$ we obtain a quadratic equation in $\frac{a_1}{a_2}$ with solutions

$$\frac{a_1}{a_2}=\frac{-pq\pm ir\sqrt{p^2+q^2+r^2}}{r^2+p^2}.$$

Analogously we obtain

$$\frac{a_3}{a_2}=\frac{-qr\pm ip\sqrt{p^2+q^2+r^2}}{r^2+p^2}.$$

Now

$$\Big|\frac{a_1}{a_2}\Big|=\frac{p^2q^2+r^2(p^2+q^2+r^2)}{(r^2+p^2)^2}=\frac{q^2+r^2}{p^2+r^2}$$

and

$$\Big|\frac{a_3}{a_2}\Big|=\frac{r^2q^2+p^2(p^2+q^2+r^2)}{(r^2+p^2)^2}=\frac{q^2+p^2}{p^2+r^2}.$$

We have

$$\frac{p-1}{q-1}=\frac{q^2+r^2}{p^2+r^2},\quad \frac{r-1}{q-1}=\frac{q^2+p^2}{p^2+r^2},$$

It follows that

$$p-r=(q-1)\frac{r^2-p^2}{p^2+r^2}\Rightarrow (p-r)\Big(1+(q-1)\frac{p+r}{p^2+r^2}\Big)=0,$$

and thus $p=r$ as $p,q,r>1$.

In the same manner we obtain $p=q=r$ and thus $|a_1|=|a_2|=|a_3|$. Now the result follows from Proposition 1. □

We may also confirm Conjecture 1 if the circle varies.

Proposition 3. *Given n different points $A_1,A_2,\ldots,A_n$ in the plane and a circle Γ such that $\sum_{i=1}^{n}|PA_i|^{2n-2}$ is a constant function of P when P belongs to an arbitrary circumference Γ_r with radius r, concentric to Γ. Then these points are the vertices of a regular polygon, inscribed in a circle concentric to Γ.*

Proof. We consider the problem in the complex plane and we assume Γ to be the unit circle. We assign complex numbers $a_1,\ldots,a_n$ to $A_1,\ldots,A_n$ respectively. Again using the complex polynomial method we obtain

$$\sum_{i=1}^{n}|x-a_i|^{2n-2}=\sum_{i=1}^{n}(x-a_i)^{n-1}(\overline{x-a_i})^{n-1}=\sum_{i=1}^{n}(x-a_i)^{n-1}(\frac{r}{x}-\overline{a_i})^{n-1}=c.$$

After multiplying out we obtain

$$(x-a_i)^{n-1}(\frac{r}{x}-\overline{a_i})^{n-1}=\sum_{j=-n+1}^{n-1}P_{ij}(r)x^j,$$

where $P_{ij}(r)$ is some polynomial in r. Now we have

$$\sum_{i=1}^{n}|x-a_i|^{2n-2}=\sum_{j=-n+1}^{n-1}\sum_{i=1}^{n-1}P_{ij}(r)x^j=c$$

and after multiplying by x^{n-1} we get:

$$\sum_{j=-n+1}^{n-1}\sum_{i=1}^{n-1}P_{ij}(r)x^{j+n-1}-cx^{n-1}=0 \tag{2.1}$$

We fix r and consider this as a polynomial in x. It has infinitely many zeros (all x with $|x|=1$) and so it is identically zero. In particularly we have that $\sum_{i=1}^{n-1}P_{ij}(r)=0$. This holds for all $r>0$, so this polynomial in r has to be identically zero. It is easy to see that for $j>0$ the leading term of P_{ij} equals $r^{n-1-j}\overline{a_i}^j$ and hence the leading term in $\sum_{i=1}^{n-1}P_{ij}(r)$ equals $\sum_{i=1}^{n-1}\overline{a_i}^j r^{n-1-j}$. From here it follows that $\sum_{i=1}^{n-1}\overline{a_i}^j=0$ for $j=1,2\dots,n-2$. It also remains to note that the leading coefficient of (2.1) as a polynomial in x is $\sum_{i=1}^{n-1}\overline{a_i}^{n-1}x^{n-1}$, hence this sum is zero and as before the proof is complete. □

3. Lemmas

Before we continue with higher dimensional analogs of the considered problems we need two auxiliary results.

The following lemma is stated and proved in [7], but nevertheless we include it here to keep the article self-contained.

Lemma 1. *Let $a_1,a_2,\dots,a_n$ and $b_1,b_2,\dots,b_n$ be real numbers and $b_i, i=1,\dots,n$ be nonnegative, then the function*

$$\Theta(\lambda)=\sum_{i=1}^{n}a_ib_i^\lambda$$

is either identically zero or has at most $n-1$ real solutions for λ counted with their multiplicities.

Proof. We proceed by induction on the number of summands. For $n=1$ we have that $ab^\lambda=0$, which does not have solutions if both of a and b are nonzero. If either of them is zero then ab^λ is identically zero. Now assume the statement to be true for all $k<n$.

For $k=n$ if either of a_i or b_i is zero then we use the induction hypothesis. Now let b_i,a_i be nonzero. As all of b_i are nonzero then we can divide each term by b_1^λ to get

$$\sum_{i=1}^{n}a_i\Big(\frac{b_i}{b_1}\Big)^\lambda=0.$$

Assume that this equation is not identically zero and its solutions are $y_1,\dots,y_k$ with multiplicities $t_1,\dots,t_k$ and $\sum_{i=1}^{k}t_i>n-1$. Differentiating this with respect to λ we get

$$\sum_{i=2}^{n}a_i\ln\Big(\frac{b_i}{b_1}\Big)\Big(\frac{b_i}{b_1}\Big)^\lambda=0=\sum_{i=2}^{n}a_i^{'}b_i^{'\lambda},$$

where $a_i'=a_i\ln\Big(\frac{b_i}{b_1}\Big)$ and $b_i'=\frac{b_i}{b_1}$. Assume that this expression is identically zero, then $\sum_{i=1}^{n}a_ib_i^\lambda=0$ must be a constant, and the claim follows. Assume that the derivative does not vanish for all λ. Now by the induction hypothesis the derivative has at most $n-2$ zeros. But we have that $y_1,\dots,y_k$ are solutions to the above equation with multiplicities $t_1-1,\dots,t_k-1$. Moreover, by Rolle's theorem the derivative has at least one root in each interval $(y_i;y_{i+1})$, and thus we obtain $k-1+\sum_{i=1}^{k}t_i-1$ solutions (counted with their multiplicities), which is greater than $n-2$- a contradiction. It follows that $\sum_{i=1}^{k}t_i\le n-1$. The lemma is proved. □

Lemma 2. *Let π and ξ be two nonparallel hyperplanes in $\mathbb{R}^n$, $n > 2$. For every two points $\mathbf{x}$ and $\mathbf{y}$ belonging to the sphere S^{n-1} there exists a sequence $S_1, \dots, S_k$ of two-codimensional spheres such that:*

(1) $\mathbf{x} \in S_1$ *and* $\mathbf{y} \in S_k$,
(2) $S_i \bigcap S_{i+1} \neq \emptyset$,
(3) $S_i = S^{n-1} \bigcap \alpha$, *where α is a hyperplane parallel either to π or to ξ.*

We shall prove more.

Lemma 3. *Let M be a C^1-smooth compact connected surface in $\mathbb{R}^n$, $n > 2$ and let $\alpha \nparallel \beta$ be hyperplanes. Then for any points $A, B \in M$ one may find points $C_0 = A, C_1, \dots, C_{n-1}, C_n = B$ on M such that $C_{i-1}C_i \parallel \alpha$ or $C_{i-1}C_i \parallel \beta$, $1 \leq i \leq n$.*

Proof. After a non-singular linear transformation, may assume that $\alpha \perp Ox_1$ and $\beta \perp Ox_2$. By the Jordan-Brouwer separation theorem, M is the boundary of a bounded domain Ω in $\mathbb{R}^n$. We may find point $D_0 = A, D_1, \dots, D_{n-1}, D_n = B \in \Omega$ such that $D_{i-1}D_i$ is parallel to a coordinate axis, $1 \leq i \leq n$. Then for any $1 \leq i \leq n$ there exist a hyperplane $\gamma_i \supset D_{i-1}D_i$ parallel to α or β. Since $D_i \in l_i \cap \Omega$, where $l_i = \gamma_i \cap \gamma_{i+1}$, one may find point $C_i \in l_i \cap M$, $1 \leq i \leq n-1$. These points have the desired property. The reason this argument fails in dimensions 1 and 2 is that the intersection of two 1 co-dimensional planes in $\mathbb{R}$ or $\mathbb{R}^2$ is discrete. □

We are now ready to continue with the Platonic solids.

4. Dodecahedron

We shall use the numbering of vertices presented in the plane projection of the dodecahedron below.

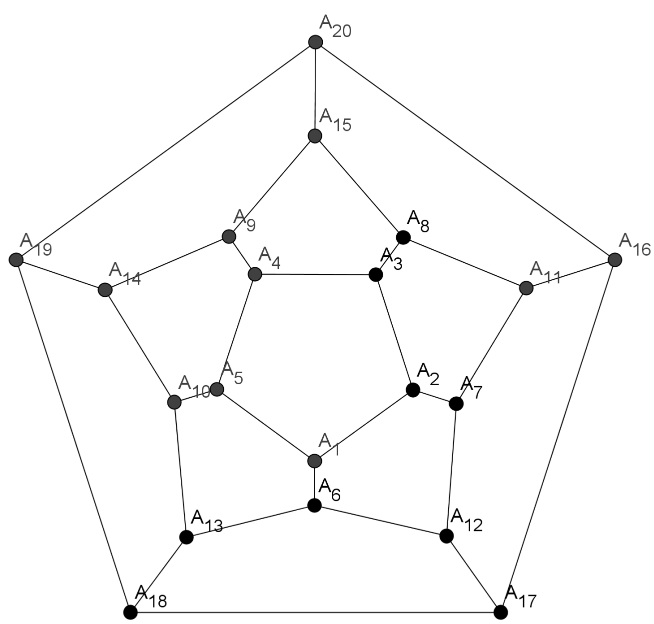

Figure 1. Plane projection of a regular dodecahedron

We investigate the following question: Given a regular dodecahedron $A_1, \dots A_{20}$ and Γ the circumscribed sphere of the polytope, determine all real numbers λ such that

$$\sum_{i=1}^{20} MA_i^{\lambda} \tag{4.1}$$

is independent of the position of M on Γ.

First we shall prove the following

Proposition 4. *There are at most eight numbers λ for which the sum $\sum_{i=1}^{20} MA_i^{\lambda}$ is independent of the position of M on Γ.*

Proof. Let $A_1A_2A_3A_4A_5$ be a face of the dodecahedron and let A be the center of the face. Define $C = OA\bigcap\Gamma$, where O is the center of Γ. Then for all numbers λ for which (4.1) is constant we have

$$\sum_{i=1}^{20} CA_i^{\lambda} = \sum_{i=1}^{20} A_1A_i^{\lambda}. \tag{4.2}$$

But we have that $|CA_i| =: d_1$ for $i \in \{1,\dots,5\}$, $|CA_i| =: d_2$ for $i \in \{6,\dots,10\}$, and so on $|CA_i| =: d_4$ for $i \in \{16,\dots,20\}$. We also have $|A_1A_i| =: d_5$ for $i \in \{6,5,2\}$, $|A_1A_i| =: d_6$ for $i \in \{3,4,7,10,13,12\}$, $|A_1A_i| =: d_7$ for $i \in \{8,9,11,14,17,18\}$ and $|A_1A_i| =: d_8$ for $i \in \{15,16,19\}$, $|A_1A_{20}| =: d_9$ (Figure 1). Then from the fact that 4.1 is a constant function of M on Γ we have

$$\sum_{i=1}^{4} 5d_i^{\lambda} = 3d_5^{\lambda} + 6d_6^{\lambda} + 6d_7^{\lambda} + 3d_8^{\lambda} + d_9^{\lambda},$$

or

$$5d_1^{\lambda} + 5d_2^{\lambda} + 5d_3^{\lambda} + 5d_4^{\lambda} - 3d_5^{\lambda} - 6d_6^{\lambda} - 6d_7^{\lambda} - 3d_8^{\lambda} - d_9^{\lambda} = 0.$$

Now it is easy to see that this cannot hold for all λ as $d_9 > d_i$ for $i \neq 9$ and we have that when $\lim_{\lambda\to\infty} 5d_1^{\lambda} + 5d_2^{\lambda} + 5d_3^{\lambda} + 5d_4^{\lambda} - 3d_5^{\lambda} - 6d_6^{\lambda} - 6d_7^{\lambda} - 3d_8^{\lambda} - d_9^{\lambda} = -\infty$. Hence by Lemma 1 we have that there are at most eight real number λ for which (4.1) is constant. □

Proposition 5. *The sum* (4.1) *is constant for* $\lambda = 2,4,6,8,10$.

Proof. Consider the plane π, which contains the points A_1, A_2, A_3, A_4, A_5, π_2, which contains the points $A_6, A_7, A_8, A_9, A_{10}$, π_3, which contains the points $A_{11}, A_{12}, A_{13}, A_{14}, A_{15}$, π_4, which contains the points $A_{16}, A_{17}, A_{18}, A_{19}, A_{20}$. First we notice that the planes π_i are parallel. Second in each of the planes π_i the points $A_{5i-4}, A_{5i-3}, A_{5i-2}, A_{5i-1}, A_{5i}$ are vertices of a regular pentagon, moreover the centers of these polygons lie on a line trough the center of the dodecahedron. Now consider a plane α, parallel to π_1 and take $\omega = \alpha\bigcap\Gamma$. Now we have that for each $M \in \omega$

$$\sum_{i=1}^{20} |MA_i|^{\lambda} = \sum_{i=1}^{5} |MA_i^{'2} + h_1^2|^{\frac{\lambda}{2}} + \sum_{i=6}^{10} |MA_i^{'2} + h_2^2|^{\frac{\lambda}{2}} + \sum_{i=11}^{15} |MA_i^{'2} + h_3^2|^{\frac{\lambda}{2}} + \sum_{i=16}^{20} |MA_i^{'2} + h_4^2|^{\frac{\lambda}{2}},$$

where h_i is the distance between the planes π_i and α and $A_i^{'}$ is the projection of the point A_i to the plane α. Now by Theorem 1 for $\lambda \in \{2,4,6,8\}$ each of the sums $\sum_{i=1}^{5} |M_1A_i^{'2} + h_1^2|^{\frac{\lambda}{2}}, \sum_{i=6}^{10} |M_2A_i^{'2} + h_2^2|^{\frac{\lambda}{2}}, \sum_{i=11}^{15} |M_3A_i^{'2} + h_3^2|^{\frac{\lambda}{2}}, \sum_{i=16}^{20} |M_4A_i^{'2} + h_4^2|^{\frac{\lambda}{2}}$ is constant.

Now for $\lambda = 10$ we have that

$$\sum_{i=1}^{5} |MA_i^{'2} + h_1^2|^5 = |MA_i^{'}|^{10} + c_4 \sum_{i=1}^{5} |MA_i^{'}|^8 + c_3 \sum_{i=1}^{5} |MA_i^{'}|^6 + c_2 \sum_{i=1}^{5} |MA_i^{'}|^4 + c_1 \sum_{i=1}^{5} |MA_i^{'}|^2 + c_0,$$

and as we know $c_j \sum_{i=1}^{5} |MA_i^{'}|^{2j}$ is constant for $j = 1,2,3,4$. Hence we only need to prove that $\sum_{i=1}^{20} |MA_i^{'}|^{10}$ is constant. But $A_1A_2A_3A_4A_5A_{16}A_{17}A_{18}A_{19}A_{20}$ and $A_6A_7A_8A_9A_{10}A_{11}A_{12}A_{13}A_{14}A_{15}$ are vertices of two regular decagons and hence by the Theorem 1 we have that (4.1) is constant for $\lambda = 10$ and $M \in \omega$.

Now by Lemma 2 this result is easily extended to the whole sphere. □

We can limit the values of λ with the desired property.

Proposition 6. *All the values of λ for which* (4.1) *is independent of the position of M on Γ are among* $2, 4 \dots, 18$.

Proof. Let π be a plane trough the center of Γ, parallel to the face $A_1A_2A_3A_4A_5$.

Let $\omega = \pi \bigcap \Gamma$ and A_i' be the projection of the point A_i in π. For $M \in \Gamma$ we have

$$\sum_{i=1}^{20} |MA_i|^\lambda = \sum_{i=1}^{5} |MA_i'^2 + h_1^2|^{\lambda/2} + \sum_{i=16}^{20} |MA_i'^2 + h_1^2|^{\lambda/2} + \sum_{i=6}^{15} |MA_i'^2 + h_2^2|^{\lambda/2},$$

where h_1 is the distance between π and the planes containing the faces $A_1A_2A_3A_4A_5$ and $A_{16}A_{17}A_{18}A_{19}A_{20}$ and h_2 is the distance between π and the planes containing the vertices $A_6, A_7, A_8, A_9, A_{10}$ and $A_{11}, A_{12}, A_{13}, A_{14}, A_{15}$. Now $A_1'A_2'A_3'A_4'A_5'A_{16}'A_{17}'A_{18}'A_{19}'A_{20}'$ and $A_6'A_7'A_8'A_9'A_{10}'A_{11}'A_{12}'A_{13}'A_{14}'A_{15}'$ are two regular homothetic dodecagons. By Theorem 1 the sum (4.1) is independent of the position of M on $\omega = \Gamma \bigcap \pi$ for $\lambda = 2, 4 \ldots, 18$ and these are the only powers with this property(using Theorem 1 for each of the decagons independently). $\square$

Remark 3. If Γ is a sphere concentric to the sphere circumscribed around the regular dodecahedron we can still consider these questions. If we apply the same approach as in the proof of Proposition 4 we can obtain that there are at most nine real λ with the desired property. Again we consider the equation (4.2), but instead of point A_1 we consider the point $A_1' = OA_1 \bigcap \Gamma$. Again using Lemma 1 the result follows. Propositions 5 and 6 still hold in this case, the proofs being analogous.

5. Icosahedron

Now we begin with the consideration of the icosahedron. We shall use the numbering of vertices presented in the plane projection of the icosahedron below.

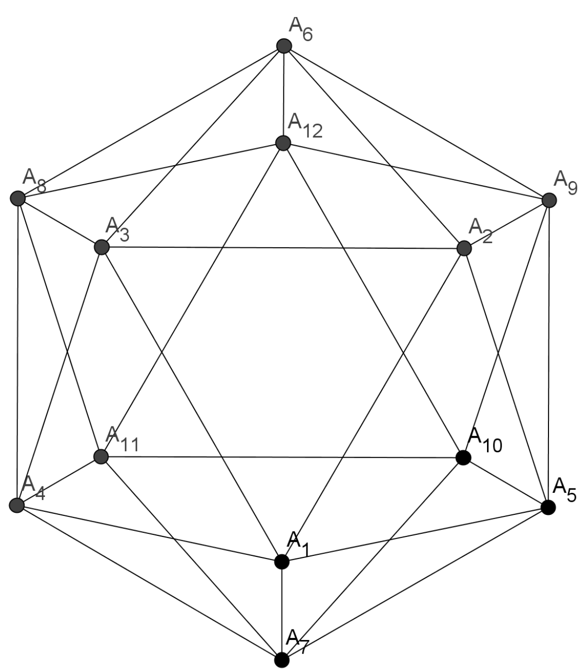

Figure 2. Plane projection of a regular icosahedron

Assume that $A_i, i \in \{1, \ldots, 12\}$ are the vertices of a regular icosahedron. Let Γ be a sphere concentric to the circumscribed sphere of the polytope. We investigate the question for which λ

$$\sum_{i=1}^{12} |A_iM|^\lambda \tag{5.1}$$

is constant for every $M \in \Gamma$.

Proposition 7. *The sum (5.1) is constant for $\lambda = 2, 4, 6$.*

Proof. Let π be a plane parallel to the plane, containing $A_1A_2A_3$ and such that $\pi\bigcup\Gamma=\omega$ is a circle. Let h_1,h_2,h_3,h_4 be the distances from the planes containing $A_1A_2A_3$, $A_4A_5A_6$, $A_7A_8A_9$ and $A_{10}A_{11}A_{12}$ to π. Obviously these planes are all parallel to π.

Now we have that

$$\sum_{i=1}^{12}|A_iM|^\lambda=\sum_{i=1}^{12}(|A_i'M|^2+h^2_{\lceil\frac{i}{3}\rceil})^{\frac{\lambda}{2}}, \tag{5.2}$$

where $M\in\omega$ and A_i' is the projection of A_i in π since we have $|A_iM|^2=|A_i'M|^2+h^2_{\lceil\frac{i}{3}\rceil}$.

Now for $\lambda=2$ we have the sum

$$\sum_{i=1}^{12}(|A_i^{'}M|^2+h^2_{\lceil\frac{i}{3}\rceil})^{\frac{\lambda}{2}}=\sum_{i=1}^{4}|MA_{3i}^{'}|^2+|MA_{3i-1}^{'}|^2+|MA_{3i-2}^{'}|^2+3h_i^2,$$

which is constant since from the planar case we know that $|MA_{3i}^{'}|^2+|MA_{3i-1}^{'}|^2+|MA_{3i-2}^{'}|^2$ is constant.

For $\lambda=4$ we have the sum

$$\sum_{i=1}^{12}(|A_i^{'}M|^2+h^2_{\lceil\frac{i}{3}\rceil})^{\frac{\lambda}{2}}=\sum_{i=1}^{4}|MA_{3i}^{'}|^4+|MA_{3i-1}^{'}|^4+|MA_{3i-2}^{'}|^4+$$
$$+2h_i^2(|MA_{3i}^{'}|^2+|MA_{3i-1}^{'}|^2+|MA_{3i-2}^{'}|^2)+3h_i^4,$$

which is constant since from the planar case we know that each of $|MA_{3i}^{'}|^4+|MA_{3i-1}^{'}|^4+|MA_{3i-2}^{'}|^4$ and $|MA_{3i}^{'}|^2+|MA_{3i-1}^{'}|^2+|MA_{3i-2}^{'}|^2$ are constant($A_1A_2A_3$, $A_4A_5A_6$, $A_7A_8A_9$ and $A_{10}A_{11}A_{12}$ all project to equilateral triangles).

For $\lambda=6$ we have

$$\sum_{i=1}^{12}(|A_i^{'}M|^2+h^2_{\lceil\frac{i}{3}\rceil})^{\frac{\lambda}{2}}=\sum_{i=1}^{12}|A_i^{'}M|^6+3h^2_{\lceil\frac{i}{3}\rceil}|A_i^{'}M|^4+3h^4_{\lceil\frac{i}{3}\rceil}|A_i^{'}M|^2+h^6_{\lceil\frac{i}{3}\rceil}.$$

As we know each of $\sum_{i=1}^{12}|A_i^{'}M|^\lambda$ is constant for $\lambda=2,4$ we need only prove that $\sum_{i=1}^{12}|A_i^{'}M|^6$ is independent on the position of M on ω. This follows directly from the consideration of the planar case and the fact that $A_1^{'}A_2^{'}A_3^{'}A_{10}^{'}A_{11}^{'}A_{12}^{'}$ and $A_4^{'}A_5^{'}A_6^{'}A_7^{'}A_8^{'}A_9^{'}$ are vertices of regular hexagons. Thus we have obtained that (5.1) is independent of the position of M on ω where ω is a circle, obtained by intersecting a plane, parallel to a plane containing a face of the icosahedron and Γ.

Now again from Lemma 2 the proposition easily follows. □

Proposition 8. *All the powers for which* (5.1) *is constant are among* $2,4\dots,10$.

Proof. Let π be a plane trough the center of Γ, parallel to the face $A_1A_2A_3$. Let $\omega=\pi\bigcap\Gamma$ and A_i' be the projection of the point A_i in π. For $M\in\Gamma$ we have

$$\sum_{i=1}^{12}|MA_i|^\lambda=\sum_{i=1}^{3}(MA_i'^2+h_1^2)^{\lambda/2}+\sum_{i=4}^{9}(MA_i'^2+h_1^2)^{\lambda/2}+\sum_{i=10}^{12}(MA_i'^2+h_2^2)^{\lambda/2},$$

where h_1 is the distance between π and the planes containing the faces $A_1A_2A_3$ and $A_{10}A_{11}A_{12}$ and h_2 is the distance between π and the planes containing the vertices A_4,A_5,A_6 and A_7,A_8,A_9. Now $A_1'A_2'A_3'A_{10}'A_{11}'A_{12}'$ and $A_4'A_5'A_6'A_7'A_8'A_9'$ are two regular homothetic hexagons. By Theorem 1 the sum (5.1) is independent of the position of M on $\omega=\Gamma\bigcap\pi$ for $\lambda=2,4\dots,10$ and these are the only powers with this property(using Theorem 1 for each of the hexagons independently). □

Remark 4. A computer check suggest that all the powers λ for which (4.1) is independent of the position of M on Γ are $\lambda=2,4\dots,10$ and all the powers λ for which (5.1) is independent of the position of M on Γ are $\lambda=2,4\dots,10$. This may be to the fact that the regular dodecahedron and the regular icosahedron are dual. No complete mathematical proof of these results is known to the authors.

6. Higher dimensional regular polytopes

Now we begin with the consideration of the higher dimensional regular polytopes. It is a known fact that in the Euclidean spaces of dimension $n > 4$ there exist only three n-dimensional regular polytopes – the regular simplex, the cross-polytope (orthoplex) and the hypercube.

We are interested in the following question: For which points are the extremal values of

$$\sum_{i=1}^{t} |A_i M|^\lambda \tag{6.1}$$

obtained, when M varies on a sphere, concentric to the sphere on which the points A_i belong when these points are the vertices of

- a regular simplex;
- a cross-polytope;
- a hypercube.

6.1. Regular simplex.

We begin with the consideration of the regular simplex. We shall prove the following:

Theorem 3. *Let $A_1 \dots A_{n+1}$ be a regular simplex in $\mathbb{R}^n$ and let Γ be a sphere concentric to the sphere circumscribed around the given simplex. Put $B_i = OA_i \bigcap \Gamma$, where O is the center of Γ and $S_n(X, \lambda) = \sum_{i=1}^{n+1} (|XA_i|^2 + h)^{\lambda/2}$, where $h > 0$ is some fixed real number. Then for any positive fixed h we have*

1. $\lambda < 0$ *Then the minimum of $S_n(X, \lambda)$ is obtained when $X = B_i O \bigcap \Gamma$ for some vertex B_i. The maximum is obtained when $X \equiv B_i$ for some B_i. When Γ is the sphere circumscribed around the regular simplex this sum is obviously unbounded ($X \to B_i$ for some B_i).*
2. $\lambda \in [0; 4]$ *If λ is an even integer, then $S_n(X, \lambda)$ is independent of the position of M on Γ. Otherwise, let $2m < \lambda < 2m + 2$. If m is even (odd), then the maximum (minimum) of $S_n(M, \lambda)$ is obtained when $X = B_i O \bigcap \Gamma$ for some i, and the minimum (maximum) is obtained when $X \equiv B_i$ for some i.*
3. $\lambda > 4$ *The maximum of $S_n(M, \lambda)$ is obtained when $X = B_i O \bigcap \Gamma$ and the minimum when $X \equiv B_i$.*

Previous work on this problem has been done by Stolarsky in [9], who obtains partial characterization of the extremal values for $\lambda \in (0; 2)$ in the case when Γ is the sphere circumscribed around the simplex.

The results obtained for the regular simplex are very similar to those obtained for the equilateral triangle.

Proof. We shall first prove part 1 of the theorem. We shall use induction on the dimension of the simplex. For the planar case we know that this is true. And that $0, 2, 4$ are the only powers for which $S_n(X, \lambda)$ is constant.

Assume now that $A_1, \dots, A_{n+1}$ are vertices of a regular simplex in $\mathbb{R}^n$ and Γ a sphere, concentric to the sphere circumscribed around the given simplex. Take now a vertex A_i and a hyperplane π, $\dim \pi = n - 1$ such that π is perpendicular to OA_i and π intersects Γ in such a way that $\dim \pi \bigcap \Gamma = \Gamma' = n - 2$. This is possible as the sphere is a compact differentiable smooth manifold. Now we have that $S_n(X, \lambda) = \sum_{j=1}^{n+1} (|MA'_j|^2 + h_j|)^{\lambda/2}$ for $M \in \Gamma'$ where A'_j is the projection of A_j in the plane π and h_j is the distance between A_j and π. We have now that A_i projects to the center of Γ', hence MA_i is constant for $M \in \Gamma'$. Moreover as the polytope is a regular simplex then $h_m = h_n$, $m, n \neq i$ and A'_j, $j \neq i$ are the vertices of a regular $(n-1)$-dimensional simplex. Hence by the induction hypothesis $S_n(X, \lambda)$ is constant for $M \in \Gamma'$ and $\lambda = 0, 2, 4$. Now from Lemma 2 it follows that $S_n(M, \lambda)$ is independent on the position of M on Γ when $\lambda = 0, 2, 4$. From the induction hypothesis it also follows that this are the only values for λ with this property.

For the other cases of Theorem 3 we again use induction.

Let $\lambda \neq 0, 2, 4$. Then as Γ is a compact set and $S_n(M, \lambda)$ is a continuous function, then there is a maximum of $S_n(M, \lambda)$. Assume that this maximum is achieved at a point N. Now consider the hyperplane π_i, which contains the $(n-1)$-dimensional simplex obtained by the vertices $A_1, \ldots, A_{i-1}, A_{i+1}, \ldots, A_n$. Now let π_i' be the hyperplane, parallel to π_i, which contains N. We consider the extremal values of $S_n(M, \lambda)|_{M \in \pi_i'}$.

We have that $|MA_i|$ is constant for $M \in \pi_i'$, so we need only consider

$$\sum_{j=1, j\neq i}^{n+1} (|MA_j|^2 + h)^{\lambda/2} = \sum_{j=1, j\neq i}^{n+1} (|M'A_j|^2 + h_j^2 + h)^{\lambda/2} = S_{n-1}(M', \lambda),$$

where M' is the projection of M in π_i and $h_m = h_n, m, n \neq i$. We have that $M' \in \Gamma'$, where Γ' is the projection of $\Gamma \bigcap \pi_i'$ in π_i. Now by the induction hypothesis we have that if O' is the projection of O in the hyperplane π_i then if the maximum of $S_{n-1}(M', \lambda)$ is obtained at M_i then

(1) $\lambda < 0$ $M_i = \overrightarrow{O_i A_j} \bigcap \Gamma'$ for some $j \neq i$. In this case if $h_j^2 + h = 0$, then Γ is the sphere circumscribed around the regular simplex and the sum $S_n(M, \lambda)$ is not bounded when $\lambda < 0$.
(2) $\lambda \in (0; 2)$ $M_i = \overrightarrow{A_j O_i} \bigcap \Gamma'$ for some $j \neq i$.
(3) $\lambda \in (2; 4)$ $M_i = \overrightarrow{O_i A_j} \bigcap \Gamma'$ for some $j \neq i$.
(4) $\lambda > 4$ $M_i = \overrightarrow{A_j O_i} \bigcap \Gamma'$ for some $j \neq i$.

Now as the global maximum of $S_n(M, \lambda)$ is obtained at N, hence we have that the maximum of $S_n(M, \lambda)|_{M \in \pi_i'}$ is also obtained at N, hence we have that the projection of N in the hyperplane π_i coincides with M_i. This remains true for all planes π_i.

It remains only to prove that the only points for the projections of which this holds are the aforementioned in the Theorem. We shall prove the following Lemma

Lemma 4. *Let $A_1 \ldots A_{n+1}$ be a regular simplex in $\mathbb{R}^n$. Let Γ be a sphere circumscribed around the simplex. For every $i = 1, \ldots, n+1$ let O_i be the projection of the center of Γ in the hyperplane π_i containing the face of the simplex, which does not contain the vertex A_i. Let X be a point on Γ. Let X_i be the projection of X in π_i. If for every i $O_i X_i$ is perpendicular to so some two codimensional face of the simplex, then either $X \equiv B_i$ for some i or $X \equiv \overrightarrow{B_i O} \bigcap \Gamma$.*

Proof. It is easy to verify this in $\mathbb{R}^3$. Assume now that the dimensional is greater than 3. We have that $O_1 X_1$ is perpendicular to so some $n-2$-dimensional face of the simplex, say the face $A_3 \ldots A_{n-1}$. Now it easily follows that $|X_1 A_i| = const$, $i = 3, \ldots, n+1$. Hence $|XA_i| = const$, $i = 3, \ldots, n+1$. Making the same considerations in π_3 we obtain that X is equidistant from some other $n-2$ face of the simplex, different than $A_2 \ldots A_{n+1}$ and hence we have that X is equidistant either from all the vertices of $A_1 \ldots A_{n+1}$ and thus $X \equiv O$, but $X \in \Gamma$ and then X is equidistant from $n-1$ vertices of the simplex. The conclusion follows easily. ☐

¿From this Lemma and the induction hypothesis the proof of Theorem 3 follows.
The proof of the minimality part of Theorem 3 is analogous. ☐

In the light of the planar case one may pose the following

Conjecture 2. *Let $A_1, \ldots, A_{n+1}$ be points in $\mathbb{R}^n$. Assume that there is a sphere Γ such that $\sum_{i=1}^{n+1} |MA_i|^\lambda$ is independent of the position of M on Γ for $\lambda = 2, 4$. Then $A_1 \ldots A_{n+1}$ are vertices of a regular simplex.*

We have already shown that this is true for $n = 2$. It can also be proved for $n = 3$.

Proposition 9. *Let A_1, A_2, A_3, A_4 be points in $\mathbb{R}^3$. Assume that there is a sphere Γ such that $\sum_{i=1}^{4} |MA_i|^\lambda$ is independent of the position of M on Γ for $\lambda = 2, 4$. Then $A_1 A_2 A_3 A_4$ is a regular tetrahedron.*

Proof. We shall present the proof here. Consider a plane π, such that $\pi \perp OA_1$ and $\pi \bigcap \Gamma = \omega$, a circle. Now we have that for $M \in \omega$ $|MA_1|$ is constant, so we need only consider the sum $|MA_2|^\lambda + |MA_3|^\lambda + |MA_4|^\lambda$. Let A_i' be the projection of the point A_i in the plane π. we have that $|MA_2|^2 + |MA_3|^2 + |MA_4|^2 = |MA_2'|^2 + h_2^2 + |MA_3'|^2 + h_3^2 + |MA_4'|^2 + h_4^2$, hence

$|MA_2'|^2+|MA_3'|^2+|MA_4'|^2$ is also constant for $M\in\omega$. We also have that $|MA_2|^4+|MA_3|^4+|MA_4|^4=|MA_2'|^4+|MA_3'|^4+|MA_4'|^4+2|MA_2'|^2h_2^2+2|MA_3'|^2h_3^2+2|MA_4'|^2h_4^2+h_2^4+h_3^4+h_4^4$. Now consider π to be the complex plane with origin-the center of ω. Assign to the points A_i' the complex numbers α_i. Using the same approach, as in the proof of the planar case we obtain that $\sum\alpha_i=0$ and $\sum\alpha_i^2=0$, from where it follows that $\alpha_i=z\xi^i$, where ξ is the third root of unity. From here it follows that A_i' are the vertices of an equilateral triangle, centered at the origin of the complex plane. Now again by the considerations at the beginning of this paper it follows that $|MA_2'|^4+|MA_3'|^4+|MA_4'|$ is constant for $M\in\omega$, hence $2|MA_2'|^2h_2^2+2|MA_3'|^2h_3^2+2|MA_4'|^2h_4$ is constant for $M\in\omega$. It follows that $\sum_{i=2}^{4}2z\xi^ih_i^2=0$. It is now easy to see that $h_2^2=h_3^2=h_4^2$ and hence $h_1=h_2=h_3$. Assume that two of the points A_i belong to different halfspaces, divided by the plane π, say A_2 and A_3. Then considering a plane π', such that the distance between π and π' ϵ, is such that π' still divides the points A_2 and A_3 and $\pi'\bigcap\Gamma=\omega'$, still a circle. Then it follows that $h_2+\epsilon=h_3-\epsilon$, which is impossible. Hence all of the points A_i, $i=2,3,4$ belong to one halfspace, and thus to one plane. From here we get that $A_2A_3A_4$ is an equilateral triangle. With the same considerations it follows that every three of the vertices of $A_1A_2A_3A_4$ are vertices of an equilateral triangle, hence it is a regular tetrahedron. □

6.2. **Cross-polytope.** Now we begin with the consideration of the cross-polytope. The vertices of this cross-polytope are of the form $(\pm1,0,\dots,0)$ and all permutations. It can also be considered as the unit sphere under the l_1 metric. It is the dual of the hypercube.

Let $A_1,\dots,A_{2n}$ be the vertices of the cross-polytope with A_i be the all-zero vector, with one in the $i-$th position for $i=1,\dots,n$ and $A_i=-A_{i-n}$ for $i>n$. Also let Γ be a sphere, concentric to the sphere circumscribed around the cross-polytope.

We are interested in the following question: For which points $M\in\Gamma$ are the extremal values of

$$\sum_{i=1}^{2n}|MA_i|^\lambda=C_n(M,\lambda) \tag{6.2}$$

are achieved.

This question has been previously considered in [9], where full characterization of (6.2) when Γ is the sphere circumscribed around the cross-polytope and $\lambda\in(0;2)$.

Put $B_i=OA_i\bigcap\Gamma$.

We shall prove the following

Theorem 4. (1) $\lambda<0$. *The maximum of $C_n(M,\lambda)$ is obtained when $M\equiv B_i$ for some i. In the case when Γ is the sphere circumscribed around $A_1\dots A_{2n}$ the sum $C_n(M,\lambda)$ is not bounded. The minimum is obtained for some vector $B=(\sum_{i=1}^n\pm B_i)/\sqrt{N}$ for any choice of plus and minus signs. Those are the points obtained by the intersection of the perpendicular from O to some face of the cross-polytope and Γ.*

(2) $\lambda\in[0;6]$. *If λ is an even number then $C_n(M,\lambda)$ is independent of the position of M on Γ. Otherwise let $2m<\lambda<2m+2$. If m is an even (odd) integer, then the maximum (minimum) of $C_n(M,\lambda)$ is obtained when M coincides with the any of the points B, defined as above, and the minimum (maximum) is achieved when $M\equiv B_i$ for some i.*

(3) $\lambda>6$ *Then the maximum of $C_n(M,\lambda)$ is obtained when M coincides with any of the vertices B_i, defined the minimum is obtained when $M\equiv B$ where B is defined as above.*

Proof. As in the case of the regular simplex we shall actually consider the more general function

$$C_n(M,\lambda)=\sum_{i=1}^{2n}(|MA_i|^2+h)^{\lambda/2}, \tag{6.3}$$

where h is some fixed positive real number and prove that the above theorem hods. We proceed by induction on the dimension of the polytope.

We have already considered the desired result in the planar case. We shall prove that for $\lambda=0,2,4,6$ the sum eq6.3 is independent on the position of M on Γ and that these are all the powers with that property. We have already verified this in the planar case and found it to be true. Now assume that it is true for all cross-polytopes of dimension smaller than n.

Now consider the n-dimensional cross-polytope. Consider a hyperplane π_i, perpendicular to A_iA_{n+i}, which intersects Γ such that the intersection is a sphere Γ' of dimension $n-2$. We shall prove that (6.3) is independent on the position of $M \in \Gamma'$ for $\lambda = 0, 2, 4, 6$ and these are the only powers with that property. We have that for $M \in \Gamma'$ $|MA_i|$ and $|MA_{n+i}|$ is constant, so we need only consider the function

$$\sum_{i=j,j\neq i,n+i}^{2n} (|MA_j|^2 + h)^{\lambda/2} = \sum_{j=1,i\neq i,n+i}^{2n} (|M'A_j|^2 + H^2 + h)^{\lambda/2}, \tag{6.4}$$

where M' is the projection of M in the hyperplane spanned by the vectors A_i, $i \neq i, n+i$. Now the projection of Γ' in that hyperplane is a sphere circumscribed around the $n-1$-dimensional cross-polytope with vertices A_i, $i \neq i, n+i$, hence by the induction hypothesis (6.4) is independent o the position of M on Γ' for $\lambda = 0, 2, 4, 6$ and those are the only such powers. Hence we have proved the desired result for $M \in \Gamma'$, now considering another plane π_j again intersecting Γ in an $(n-2)$-dimensional sphere Γ'_j and perpendicular to A_jA_{n+j}. Again with the same argument we obtain that the sum (6.3) is independent on the position of M on Γ'_j for $\lambda = 0, 2, 4, 6$ and those are the only powers with this property.. Now using Lemma 2 it follows that this result can be extended to the whole sphere Γ.

We now proceed with the consideration of the extremal points. We shall only consider the case or $\lambda < 0$ as the desired results in other cases of Theorem 4 can be obtained in the same manner.

We shall apply the same approach we used with the regular simplex. We again use induction on the dimension of the cross-polytope. Now let $\lambda < 0$. Assume that the minimum of (6.2) is obtained at some point N on Γ. Take a hyperplane π_i, containing N and parallel to the hyperplane σ_i spanned by the vectors A_j, $i \neq j, j+n$. Put $\pi_i \bigcap \Gamma = \Gamma_i$. We have that $\dim \Gamma_i$ is either zero (in that case N coincides with some vertex of the cross-polytope) or $\dim \Gamma_i = n-2$. There is at most one non-parallel hyperplane for which the first case occurs. Take i, such that $\dim \Gamma_i = n-2$. We have that $|MA_i|$ and $|MA_{i+n}|$ are constant for $M \in \Gamma_i$, so we need only consider the sum $\sum_{j=1,j\neq i,i+n}^{2n} |MA_j|^\lambda$.

Now we consider

$$\sum_{j=1,j\neq i,i+n}^{2n} (|M'A_j|^2 + H + h)^{\lambda/2}, \tag{6.5}$$

where H is the squared distances between the planes π_i and σ_i and M' is the projection of M in the plane σ_i. We have now that M' belongs to the projection of Γ_i in the plane σ_i, which is a $(n-2)$-dimensional sphere, concentric to the sphere circumscribed around the $(n-1)$-dimensional cross-polytope with vertices A_j, $j \neq i, i+n$.

Now as the global minimum of $C_n(M, \lambda)$ is obtained at N, so a local minimum must be obtained at that point, hence N' is the point for which (6.5) achieves its minimum.

We consider the $n-1$-dimensional subspace, the hyperplane σ_i. From the induction hypothesis and the fact that N' is the point for which the sum (6.5) achieves its minimum it follows that the coordinates of N' in that space are given by $c\mathbf{v}$, where c is a constant and $\mathbf{v}$ is the vector, each coordinate of which is ± 1. Hence the j-th, $j \neq i$ coordinate of N is $\pm c$ and the i-th coordinate of N is $\pm\sqrt{H}$. Now consider another hyperplane π_j, $j \neq i$ defined in the same way as π_i (as $n \geq 3$ such plane exists), with analogous considerations it follows that the $k-$th, $k \neq j$ coordinate of the point N is $\pm s$, where s is a constant. From here it follows that the coordinates of N are all $\pm a$, where a is a constant. Taking into consideration that N lies on the sphere Γ, the desired result follows.

Now assume that Γ is not the sphere circumscribed around $A_1 \ldots A_{2n}$, as in that case (6.3) is not bounded when $M \to A_i$ for some i. Assume that the maximum of (6.3) is achieved at a point T. As before we take a hyperplane π_i, containing T and parallel to the hyperplane σ_i spanned by the vectors A_j, $i \neq j, j+n$. Put $\pi_i \bigcap \Gamma = \Gamma_i$. We have that $\dim \Gamma_i$ is either zero-in that case N coincides with some vertex of the cross-polytope and we have nothing to prove.

Assume that this is not the case, then $\dim \Gamma_i = n-2$. We have that $|MA_i|$ and $|MA_{i+n}|$ are constant for $M \in \Gamma_i$, so we need only consider the sum $\sum_{j=1, j\neq i, i+n}^{2n} |MA_j|^{\lambda}$.

As before consider the projection of the point M in the hyperplane σ_i, spanned by the vectors A_j, $j \neq i, i+n$. Then again by the induction hypothesis it follows that if T projects in T', then in the subspace σ_i T' has at most one nonzero coordinate, let it be the k-th coordinate. It follows that T has at most two nonzero coordinates. As T belongs to Γ, T cannot coincide with the origin of the coordinate space. If it has one nonzero coordinate, the conclusion follows. Now assume that it has two nonzero coordinates, say k and i. Take now another hyperplane π_j, $j \neq k, i$, defined in the same fashion as π_i, as $n \geq 3$ such a hyperplane exists. Applying the same methodology it follows that at it is not possible both the k-th and the i-th coordinate to be nonzero, hence T has exactly one nonzero coordinate. Taking into consideration that $|T|$ is equal to the radius of Γ the desired result follows. □

We may consider the following

Conjecture 3. *Let $A_1, \ldots, A_{2n}$ be different points in $\mathbb{R}^n$. Assume that there is an sphere Γ with $\dim \Gamma = n-1$, such that $\sum_{i=1}^{n+1} |MA_i|^{\lambda}$ is independent of the position of M on Γ for $\lambda = 2, 4, 6$, then $A_1, \ldots, A_{2n}$ are the vertices of an n-dimensional cross-polytope.*

While this is true for the planar case, even the three-dimensional case is very difficult and the authors do not have a proof for $n > 2$.

6.3. **Hypercube.** We now begin with the consideration of the hypercube. For ease of the introduction we shall consider the standard unit cube under the translation with the vector $\mathbf{v} = (-1/2, \ldots, -1/2)$, hence all the vertices of the cube have coordinates $(\pm 1/2, \ldots, \pm 1/2)$.

We again investigate the extremal properties of the sum

$$H_n(M, \lambda) = \sum_{i=1}^{2^n} (|MA_i|^2 + h)^{\lambda/2}, \tag{6.6}$$

where A_i are the vertices of the n-dimensional hypercube in $\mathbb{R}^n$, and M is a point on a sphere Γ, concentric to the sphere circumscribed around the cube.

Previous work on this topic includes [9], where partial results have been obtained f or the case in which Γ is the sphere circumscribed around the polytope and $\lambda \in (0; 2)$.

Let $\mathbf{e}_i$ be the standard orthonormal vector base for $\mathbb{R}^n$, and r the radius of Γ. Put $B_i O A_i \bigcap \Gamma$, where O is the center of Γ.

We shall prove the following theorem

Theorem 5.
1. $\lambda < 0$ *The maximum of $H_n(M, \lambda)$ is obtained when $M = B_i$ for some i. In the case when Γ is the sphere circumscribed around $A_1 \ldots A_{2^n}$ the sum $H_n(M, \lambda)$ is not bounded. The minimum is obtained for the point corresponding to the vector $r\mathbf{e}_i$ for some i. Those are the points obtained by the intersection of the perpendicular from O to some $n-1$face of the cube and Γ.*
2. $\lambda \in [0; 6]$. *If λ is an even number, then $H_n(M, \lambda)$ is independent of the position of M on Γ. Otherwise let $2m < \lambda < 2m+2$. If m is an even (odd) integer, then the maximum (minimum) of $H_n(M, \lambda)$ is obtained when M is the point corresponding to some vector $r\mathbf{e}_i$,. The minimum(maximum) is achieved when $M \equiv B_i$ for some i.*
3. $\lambda > 6$ *Then the maximum of $H_n(M, \lambda)$ is obtained when M coincides with a vertex B_i, and the minimum is obtained when M is the point corresponding to some vector $r\mathbf{e}_i$.*

The points B_i can also be characterized by the vectors $\frac{r}{\sqrt{n}}(\pm 1/2, \ldots, \pm 1/2)$, and this characterization we shall use in the proof.

The results for the hypercube are very similar to those obtained for the cross-polytope. One may attribute this to the polytopes being dual. It is worth exploring similarities in the properties of functions of the type of (6.6) for dual polytopes.

Proof. We shall first prove that (6.6) is independent on the position of M on Γ for $\lambda = 0, 2, 4, 6$, and that these are the only powers for which this is true.

We shall proceed as before with induction on the dimension of the cube. The result has already been proved for the two-dimensional case. Assume it to be true for all dimensions less than n.

Let π_i and π_i' be two planes parallel to the hyperplane defined by $x_i = 0$ and containing all of the vertices of the cube (each hyperplane contains one $n-1$ face of the cube). Let now σ_i, be a hyperplane, parallel to π_i, such that $\dim \sigma_i \bigcap \Gamma = \Gamma_i = n-2$. We shall prove that (6.6) is independent on the position of M on Γ_i for $\lambda = 0, 2, 4, 6$.

Let the $n-1$ face of the hypercube, contained in π_i be S and the $n-1$ face contained in π_i' be S'. The vertices of each of S and S' are themselves vertices of n-dimensional cubes. Now we have

$$\begin{aligned} H_n(M,\lambda)|_{M\in\Gamma_i} = \sum_{i=1}^{2^n}(|MA_i|^2+h)^{\lambda/2} = \sum_{A_i\in S}(|M_1A_i|^2+H_1+h)^{\lambda/2} + \\ \sum_{A_i\in S'}(|M_2A_i|^2+H_2+h)^{\lambda/2}, \end{aligned} \tag{6.7}$$

where M_1 is the projection of the point M in the plane π_i, H_1 is the squared distance between the planes π_i and σ_i, M_2 is the projection of M in π_i', and H_2 is the squared distance between the planes π_i' and σ_i. Now as the as the vertices $A_i \in S$ and $A_j \in S'$ are vertices of lower dimensional cubes H_1 and H_2, and Γ_i projects in each of the planes π_i and π_i' to spheres concentric to the spheres circumscribed around H_1 and H_2 from the inductive hypothesis it follows that each of the sums $\sum_{A_i\in S}(|M_1A_i|^2+H_1+h)^{\lambda/2}$, $\sum_{A_i\in S'}(|M_2A_i|^2+H_2+h)^{\lambda/2}$ is independent of the position of M in Γ_i for $\lambda = 0, 2, 4, 6$. Taking another two planes π_j and π_j', $i \neq j$, defined in the same fashion as π_i and π_i' and with the same considerations it follows that 6.6 is also independent of the position of M on Γ_j (defined in the same way as Γ_i) for $\lambda = 0, 2, 4, 6$. Now by Lemma 2 it follows that this can be extended to the whole sphere Γ. To prove that $\lambda = 0, 2, 4, 6$ are the only powers with that property we need only consider a plane σ_i, defined by $x_i = 0$, then taking π_i and π_i' as before, we have

$$\begin{aligned} H_n(M,\lambda)|_{M\in\Gamma_i} = \sum_{i=1}^{2^n}(|MA_i|^2+h)^{\lambda/2} = \sum_{A_i\in S}(|M_1A_i|^2+H_1+h)^{\lambda/2} + \\ + \sum_{A_i\in S'}(|M_2A_i|^2+H_2+h)^{\lambda/2} = 2\sum_{A_i\in S}(|M_1A_i|^2+H_1+h)^{\lambda/2}, \end{aligned} \tag{6.8}$$

due to symmetry and the desired result follows immediately from the induction hypothesis, as M_1 belongs to sphere concentric to the sphere circumscribed around the $n-1$-dimensional cube H_1.

Now we shall prove the extremal cases of Theorem 5.

We will only consider the case $\lambda < 0$ as other cases can be proved in the same way. We proceed by induction. The have already proved the claim for planar case.

Assume that Theorem 5 is true and assume that the maximum of 6.6 is achieved at some point N. We shall prove that $N = (\pm c, \dots, \pm c)$. Consider a plane σ_i, parallel to the plane $x_i = 0$ and as above consider the planes π_i and π_i'. Suppose that $\dim \sigma_i \bigcap \Gamma = \Gamma_i = n-2$, this is possible as for every point N there is exactly one class of nonparallel hyperplanes such that $\dim \sigma_i \bigcap \Gamma = \Gamma_i = 0$ and in the case we consider $n \geq 3$. Now again we consider $H_n(M,\lambda)|_{\Gamma_i}$. As the global maximum of (6.6) is achieved in N, so it is the local, hence $\max H_n(M,\lambda)|_{\Gamma_i} = H_n(N,\lambda)|_{\Gamma_i}$ for a fixed $\lambda < 0$.

We again have the identity (6.8) holds, and due to symmetry and the induction hypothesis it follows that N_1 (the projection of N in π_i) is the point for which the sum $\sum_{A_i\in S}(|M_1A_i|^2+H_1+h)^{\lambda/2}$ is maximized and N_2 (the projection of N in π_i') the point for which $\sum_{A_i\in S'}(|M_2A_i|^2+H_2+h)^{\lambda/2}$ is maximized. Hence again by the induction hypothesis it follows that the all the coordinates of N, except the $i-th$ are of the form $\pm c_0$. Taking another plane σ_j, $j \neq i$, defined in the same fashion (this is possible as $n \geq 3$) and repeating the above reasoning we obtain that

all the coordinates of N, except the $j-$th are of the form $\pm c_1$. Now as $n \geq 3$ it follows that all coordinates of N are $\pm c$, taking into account that $|ON| = r$ the conclusion follows.

Assume that the minimum of (6.6) is obtained for a point T. We now begin with the proof of the minimal case. Again we consider a plane σ_i, parallel to the plane $x_i = 0$ containing T. Assume that $\dim \sigma_i \bigcap \Gamma = \Gamma_i = 0$, and in that case we have the desired result $N = \mathbf{e}_i$. Assume now that $\dim \sigma_i \bigcap \Gamma = \Gamma_i = n - 2$.

Now we again consider $H_n(M, \lambda)|_{\Gamma_i}$, for which (6.8) holds, and again by symmetry and the induction hypothesis we have that T_1 (the projection of T in π_i) is the point for which the sum $\sum_{A_i \in S} (|M_1 A_i|^2 + H_1 + h)^{\lambda/2}$ is minimized and T_2 (the projection of T in π_i') the point for which $\sum_{A_i \in S'} (|M_2 A_i|^2 + H_2 + h)^{\lambda/2}$ is minimized. It follows that among all the coordinates of N, except the $i-$th there is at most one nonzero, say the $j-$th. Now if we again consider a hyperplane σ_k, $k \neq i, j$ defined in the same fashion as σ_i (this is possible due to $n \geq 3$ and the consideration of $\dim \sigma_i \bigcap \Gamma = \Gamma_i = 0$ done in the beginning) and repeating the above considerations it follows that among all coordinates of T, except the $k-$th there is at most one nonzero. Now we easily get that N has at most one nonzero coordinate. As we have that $|OT| = r$ we obtain the desired result. ☐

Another question which we can consider is the following: Let $A_1, \ldots, A_{2^n}$ be different points in $\mathbb{R}^n$. Assume that there is a sphere Γ with $\dim \Gamma = n - 1$, such that $\sum_{i=1}^{2^n} |MA_i|^\lambda$ is independent of the position of M on Γ for $\lambda = 2, 4, 6$, then $A_1 \ldots A_{2^n}$ are the vertices of an n-dimensional cube.

But this is not generally true. In $\mathbb{R}^n$, n being a power of two greater than first, we can consider $2^n/2n$ different rotations under which the images of no two points coincide of the cross-polytope. The set of points obtained in this way satisfies the condition. No counterexamples are known to the authors in different dimensions.

One can see many similarities between the cases with the cross-polytope and the cube. We have proved the following

Take the standard cross-polytope $A_1 \ldots A_{2n}$ and the unit cube described in the beginning of this section $B_1 \ldots B_{2^n}$, and consider each of the sums (6.2) and (6.6). Assume that for some λ the minimum of (6.2) is obtained at some point N, then the maximum of (6.6) is obtained at N and vice versa. This may be to the fact of the duality of the two polytopes. Here based on the results in the planar case, the results for the regular simplex, and the dual pair cube, cross-polytope for which it holds, we can formulate the following:

Conjecture 4. *Let P be an n-dimensional regular polytope, in $\mathbb{R}^n$, inscribed in the unit sphere U. Let S be the polytope obtained under the polar reciprocation of P in U. Put P' be a homothetic polytope of S, inscribed in U. Let $\mathbf{V}$ be the set of vertices of P, and $\mathbf{V}'$ the set of vertices of P'. Consider the sums*

$$\sum_{A \in \mathbf{V}} |MA|^\lambda, \tag{6.9}$$

$$\sum_{A \in \mathbf{V}'} |MA|^\lambda, \tag{6.10}$$

where M belongs to some sphere concentric with the units sphere. Assume that

the maximum of (6.9) *is obtained for some point N, then the minimum of* (6.10) *is obtained at N, and vice versa. If* (6.9) *is independent of the movement of M on Γ then so is* (6.10) *and vice versa.*

Trough the results proved so far we have verified this proposition for the regular polygons, regular simplex, and the dual pair cube, cross-polytope. It remains to be proved for the dual pairs icosahedron and the dodecahedron, the 600-cell and the 120- cell in 4-dimensional space.

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